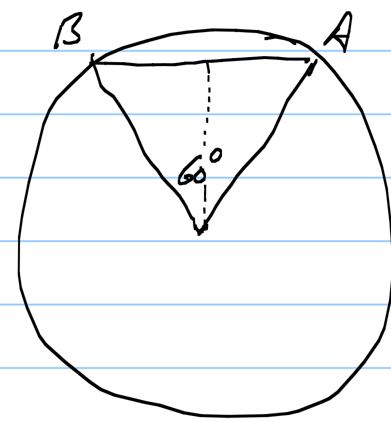
Chapter 16 - Wave Motion

Note Title 5/22/2006

5. Two points A and B on the surface of the Earth are at the same longitude and 60.0° apart in latitude. Suppose that an earthquake at point A creates a P wave that reaches point B by traveling straight through the body of the Earth at a constant speed of 7.80 km/s. The earthquake also radiates a *Rayleigh wave*, which travels across the surface of the Earth in an analogous way to a surface wave on water, at 4.50 km/s.

(a) Which of these two seismic waves arrives at B first? (b) What is the time difference between the arrivals of the two waves at B? Take the radius of the Earth to be 6 370 km.



(a) Longitudinal wave travels directly $\overline{AB}$ and is faster, so it arrives first.

(b) $AB = 2(R\sin 30^\circ) = R$

$$\therefore T = \frac{R}{7.80\times10^3\ m/sec} = \frac{6.87\times10^6\ m}{7.80\times10^3\ m/sec} = 817\ sec$$

$$d\ \overset{\frown}{AB} = R\theta = 6.37\times10^6\ m\left(\tfrac{\pi}{3}\ rad\right) = 6.67\times10^6\ m$$

$$\therefore T = \frac{6.67\times10^6\ m}{4.50\times10^3\ m/sec} = 1480\ sec$$

$\therefore 1480 - 817 = \underline{\underline{666\ sec}}$ (rounding from calc).

6. S and P waves, simultaneously radiated from the hypocenter of an earthquake, are received at a seismographic station 17.3 s apart. Assume the waves have traveled over the same path at speeds of 4.50 km/s and 7.80 km/s. Find the distance from the seismograph to the hypocenter of the quake.

$$\frac{d}{4.5} - \frac{d}{7.8} = 17.3, \quad (7.8-4.5)d = 17.3(4.5)(7.8)$$

$$d = \frac{(17.3)(4.5)(7.8)}{7.8-4.5} = \underline{184\text{ km}}$$

7.] Two sinusoidal waves in a string are defined by the functions

$$y_1 = (2.00\text{ cm})\sin(20.0x - 32.0t)$$

and

$$y_2 = (2.00\text{ cm})\sin(25.0x - 40.0t)$$

where y_1, y_2, and x are in centimeters and t is in seconds. (a) What is the phase difference between these two waves at the point $x = 5.00$ cm at $t = 2.00$ s? (b) What is the positive x value closest to the origin for which the two phases differ by $\pm\pi$ at $t = 2.00$ s? (This is where the two waves add to zero.)

(a) Not clear what "phase difference" means. Presumably just looking at y values.

$\therefore$ at $x=5, t=2$, $y_1: 20(5) - 32(2) = 36$ rad
$y_2: 25(5) - 40(2) = 45$ rad

$\therefore y_2$ is 9 rad ahead of y_1.

(b) From (a) $\Delta\phi = 25x - 40t - (20x - 32t)$

$\therefore \Delta\phi = 5x - 8t$

$\sin(\theta + 2n\pi) = \sin\theta$

$\therefore$ for $t = 2$, solve for $\Delta\phi = 5x - 16 + 2n\pi$,
where $\Delta\phi = \pm\pi$

$\therefore \pm\pi = 5x - 16 + 2n\pi$
$\therefore 5x = 16 - 2n\pi \pm \pi$

When x is as small as possible,
$16 - 2n\pi \pm \pi$ is as small as possible.
$\therefore$ Look at $16 - 2n\pi + \pi$
and $16 - 2n\pi - \pi$
These are the same for n integer.
$\therefore$ Look at $16 - 5\pi = 0.292$
and $16 - 6\pi = 2.85$

$\therefore 5x = 0.292$, $\underline{x = 0.0584 \text{ cm}}$

9. Two pulses traveling on the same string are described by

$$y_1 = \frac{5}{(3x - 4t)^2 + 2} \qquad \text{and} \qquad y_2 = \frac{-5}{(3x + 4t - 6)^2 + 2}$$

(a) In which direction does each pulse travel? (b) At what time do the two cancel everywhere? (c) At what point do the two pulses always cancel?

(a) $-4t \Rightarrow y_1$ travels to right.

$+4t-6 \Rightarrow y_2$ travels to left.

(b) To cancel everywhere for a time t,

$3x - 4t = 3x + 4t - 6$, or $6 = 8t$,

$t = 3/4$ sec.

(c) $(3x - 4t)^2 = (3x + 4t - 6)^2$

For $3x - 4t = 3x + 4t - 6$, There is no x.

For $-(3x - 4t) = 3x + 4t - 6$

$-3x + 4t = 3x + 4t - 6$

$6 = 6x$, $\underline{x = 1\,cm}$

10. A telephone cord is 4.00 m long. The cord has a mass of 0.200 kg. A transverse pulse is produced by plucking one end of the taut cord. The pulse makes four trips down and back along the cord in 0.800 s. What is the tension in the cord?

$4(8.00\,m) = 32.0\,m$, $\frac{32}{0.8} = 40\,m/sec.$

$v = \sqrt{\frac{T}{\mu}}$, $T = \mu v^2 = \frac{(0.20\,kg)}{4.00\,m}(40^2\,m^2/sec^2) = 80.0\,N$

11. Transverse waves with a speed of 50.0 m/s are to be produced in a taut string. A 5.00-m length of string with a total mass of 0.060 0 kg is used. What is the required tension?

$$\mu = \frac{0.060\ Kg}{5.00\ m} = 0.012\ Kg/m$$

$$V = \sqrt{\frac{T}{\mu}} \Rightarrow T = \mu v^2$$

$$= (0.012\ Kg/m)(50.0\ m/sec)^2$$

$$= 30.0\ N$$

12. A piano string having a mass per unit length equal to 5.00×10^{-3} kg/m is under a tension of 1 350 N. Find the speed of a wave traveling on this string.

$$V = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{1.35 \times 10^{3}}{5.00 \times 10^{-3}}} = \sqrt{0.27 \times 10^{6}} = \underline{\underline{520\ m/sec}}$$

13. An astronaut on the Moon wishes to measure the local value of the free-fall acceleration by timing pulses traveling down a wire that has an object of large mass suspended from it. Assume a wire has a mass of 4.00 g and a length of 1.60 m, and that a 3.00-kg object is suspended from it. A pulse requires 36.1 ms to traverse the length of the wire. Calculate g_{Moon} from these data. (You may ignore the mass of the wire when calculating the tension in it.)

$F_g = \frac{G M_m m_o}{r_m^2}$, $\frac{F_g}{m_o} = \frac{G M_m}{r_m^2}$, so $g_m = \frac{G M_m}{r_m^2}$

where r_m = radius of moon, M_m = moon mass.

$\mu_{wire} = \frac{0.004\ kg}{1.6\ m} = 2.5 \times 10^{-3}\ kg/m$

$T = (g_m)(3.00\ kg)$ $\quad V = \frac{1.60\ m}{36.1 \times 10^{-3}\ sec} = 44.3\ m/sec$

$\therefore V^2 \mu = T = 3.00\, g_m$, $g_m = \frac{(44.3)^2 (2.5 \times 10^{-3})}{3.00}$

$\therefore g_m = \underline{1.64\ m/sec^2}$

16. A simple pendulum consists of a ball of mass M hanging from a uniform string of mass m and length L, with $m \ll M$. If the period of oscillations for the pendulum is T, determine the speed of a transverse wave in the string when the pendulum hangs at rest.

For a pendulum, $T = 2\pi \sqrt{\frac{L}{g}}$

$\therefore L = \frac{g T^2}{4\pi^2}$. $\mu = \frac{m}{L}$. Tension $= Mg$

$$\therefore\ v = \sqrt{\frac{\text{Tension}}{\mu}} = \sqrt{\frac{mg}{m/L}} = \sqrt{\frac{gML}{m}}$$

$$= \sqrt{\frac{gM}{m}\left(\frac{gT^2}{4\pi^2}\right)} = \underline{\frac{gT}{2\pi}\sqrt{\frac{M}{m}}}$$

21. A 30.0-m steel wire and a 20.0-m copper wire, both with 1.00-mm diameters, are connected end to end and are stretched to a tension of 150 N. How long does it take a transverse wave to travel the entire length of the two wires?

Speed is different in each wire.

$\therefore$ Find time of travel in each wire & add.

$$t = \frac{L}{v} = \frac{L}{\sqrt{\frac{T}{\mu}}} = L\sqrt{\frac{\mu}{T}}.$$

$$\mu = \frac{m}{L},\quad \rho = \frac{m}{Vol} = \frac{m}{AL},\quad \therefore\ m = \rho A L$$

$$\therefore\ \mu = \frac{\rho A \cancel{L}}{\cancel{L}} = \rho A$$

$$A = \frac{\pi d^2}{4},\ \text{so}\ \mu = \frac{\rho \pi d^2}{4}$$

$$\therefore\ t = L\sqrt{\frac{\rho \pi d^2}{4T}} = \frac{Ld}{2}\sqrt{\frac{\rho \pi}{T}}$$

$\rho_{Cu} = 8.92 \times 10^3 \ Kg/m^3$ (p. 9, p. 461)

$\rho_{steel} \simeq \rho_{Fe} = 7.86 \times 10^3 \ Kg/m^3$

$$\therefore t_{Cu} = \frac{20(1.0 \times 10^{-3})}{2} \sqrt{\frac{8.92 \times 10^3 (\pi)}{150}} = 0.137 \text{ sec}$$

$$t_{steel} = \frac{30(1.0 \times 10^{-3})}{2} \sqrt{\frac{7.86 \times 10^3 (\pi)}{150}} = 0.192 \text{ sec}$$

$$\therefore t_{total} = 0.137 + 0.192 = \underline{0.329 \text{ sec.}}$$

25. A sinusoidal wave is traveling along a rope. The oscillator that generates the wave completes 40.0 vibrations in 30.0 s. Also, a given maximum travels 425 cm along the rope in 10.0 s. What is the wavelength?

$$v = \lambda f, \quad \lambda = \frac{v}{f} = \frac{425 \text{cm}/10 \text{sec}}{40/30 \text{ sec}} = \underline{31.9 \text{ cm}}$$

28. A sinusoidal wave traveling in the $-x$ direction (to the left) has an amplitude of 20.0 cm, a wavelength of 35.0 cm, and a frequency of 12.0 Hz. The transverse position of an element of the medium at $t = 0$, $x = 0$ is $y = -3.00$ cm, and the element has a positive velocity here. (a) Sketch the wave at $t = 0$. (b) Find the angular wave number, period, angular frequency, and wave speed of the wave. (c) Write an expression for the wave function $y(x, t)$.

$$y(x,t) = A\sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) + \phi\right)$$

$A = 20.0\text{ cm}$

When $x=0$, $t=0$, $y(0,0) = -3.00\text{ cm}$

$\therefore -3.00 = 20\sin(\phi)$, $-0.15 = \sin\phi$, $\phi = -0.151$

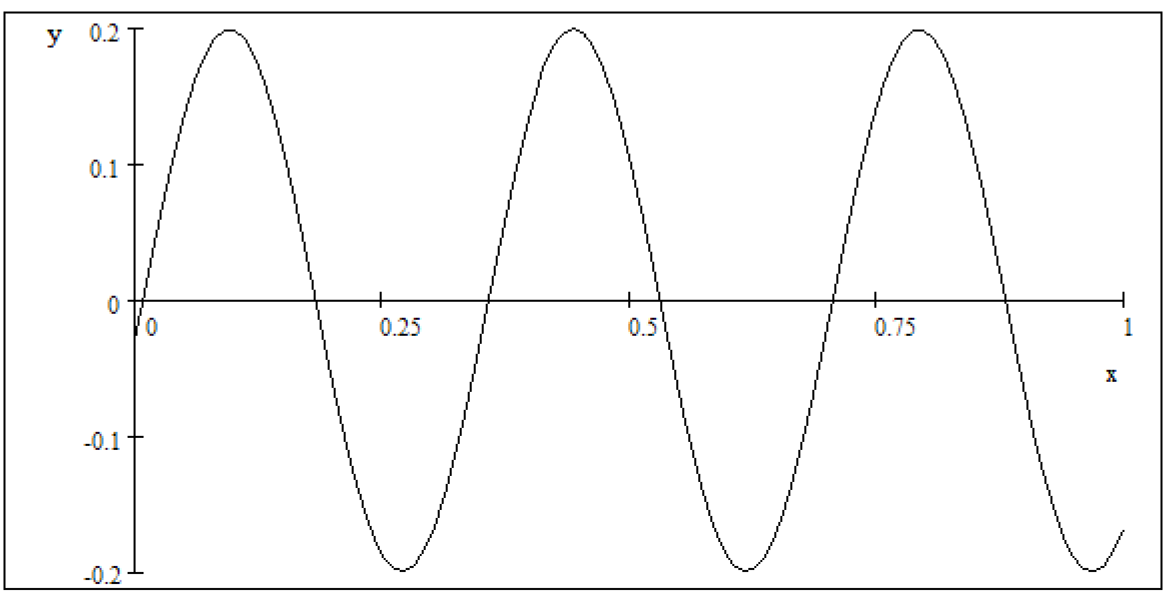
(c) $y(x,t) = 20\sin\left(\frac{2\pi}{35}x - 24\pi t - 0.151\right)$, x in cm

or,

$y(x,t) = 0.20\sin(18.0x - 24\pi t - 0.151)$, x in m.

(b) $K = \frac{2\pi}{\lambda} = \frac{2\pi}{0.35} = 18.0$ rad/m

(a)



29. A sinusoidal wave is described by

$$y = (0.25\text{ m})\sin(0.30x - 40t)$$

where x and y are in meters and t is in seconds. Determine for this wave the (a) amplitude, (b) angular frequency, (c) angular wave number, (d) wavelength, (e) wave speed, and (f) direction of motion.

(a) Amplitude = 0.25 m

(b) ω = 40 rad/sec.

(c) $K = 0.30$ rad/m

(d) $K = \frac{2\pi}{\lambda}$, $\lambda = \frac{2\pi}{K} = \frac{2\pi}{0.30} = 20.9\text{ m}$

(e) $\omega T = K\lambda$, $\omega = \frac{K\lambda}{T} = K\lambda f = KV$, $V = \frac{\omega}{K} = \frac{40}{0.3}$

$V = 133$ m/sec

(f) $-40t \Rightarrow$ direction to right

30. A transverse wave on a string is described by the wave function

$$y = (0.120\text{ m})\sin[(\pi x/8) + 4\pi t]$$

(a) Determine the transverse speed and acceleration at $t = 0.200$ s for the point on the string located at $x = 1.60$ m. (b) What are the wavelength, period, and speed of propagation of this wave?

(a) $V_y = \frac{\partial y(x,t)}{\partial t} = (0.120\text{ m})(4\pi)\cos[(\pi x/8) + 4\pi t]$

$a_y = \frac{\partial^2 y(x,t)}{\partial t^2} = -(0.120\text{ m})(16\pi^2)\sin[(\pi x/8) + 4\pi t]$

$\therefore$ at $x = 1.60\,m$ and $t = 0.200\,sec$,

$$V_y = (0.120)(4\pi)\cos[0.2\pi + 0.8\pi]$$

$$= -(0.120)(4\pi) = \underline{-1.51\,m/sec}$$

$a_y = 0$ since $\sin[0.2\pi + 0.8\pi] = 0$

(b) $K = \frac{\pi}{8} = \frac{2\pi}{\lambda}$, $\therefore \lambda = 16.0\,m$

$\omega = 4\pi = \frac{2\pi}{T}$, so $T = 0.5\,sec$

$v = \lambda f = \frac{\lambda}{T} = \frac{16.0}{0.5} = 32\,m/sec.$

31. (a) Write the expression for y as a function of x and t for a sinusoidal wave traveling along a rope in the *negative* x direction with the following characteristics: $A = 8.00$ cm, $\lambda = 80.0$ cm, $f = 3.00$ Hz, and $y(0, t) = 0$ at $t = 0$. (b) **What If?** Write the expression for y as a function of x and t for the wave in part (a) assuming that $y(x, 0) = 0$ at the point $x = 10.0$ cm.

(a) $K = \frac{2\pi}{\lambda} = \frac{2\pi}{0.8} = \frac{5\pi}{2}$ $\omega = 2\pi f = 6\pi$, $\phi = 0$

$$\therefore y(x,t) = 0.080\sin\left(\frac{5\pi}{2}x + 6\pi t\right)$$

(b) $0 = 0.080 \sin\left(\frac{5\pi}{2}(0.1) + \phi\right)$, $0 = \sin\left(\frac{\pi}{4} + \phi\right)$

$\therefore \phi = -\frac{\pi}{4} = -0.785$

$\therefore y(x,t) = 0.080 \sin\left(\frac{5\pi}{2} x + 6\pi t - \frac{\pi}{4}\right)$

36. A transverse traveling wave on a taut wire has an amplitude of 0.200 mm and a frequency of 500 Hz. It travels with a speed of 196 m/s. (a) Write an equation in SI units of the form $y = A \sin(kx - \omega t)$ for this wave. (b) The mass per unit length of this wire is 4.10 g/m. Find the tension in the wire.

(a) $f = 500$, $\omega = 2\pi f = 1000\pi$ rad/sec. $= 3140$ rad/sec

$V = \lambda f$, $\lambda = \frac{V}{f}$, $K = \frac{2\pi}{\lambda}$, $K = \frac{2\pi f}{V} = \frac{1000\pi}{196}$

$\therefore K = 16.0$ rad/m

$\therefore y(x,t) = 0.000200 \sin(16.0x - 3140t)$ m

(b) $V = \sqrt{\frac{T}{\mu}}$, $T = \mu V^2 = (4.10 \times 10^{-3})(196)^2$

$= 158$ N

38. A taut rope has a mass of 0.180 kg and a length of 3.60 m. What power must be supplied to the rope in order to generate sinusoidal waves having an amplitude of 0.100 m and a wavelength of 0.500 m and traveling with a speed of 30.0 m/s?

$$P = \frac{1}{2}\mu\omega^2 A^2 v \quad , \quad \mu = \frac{0.180}{3.6} = 5.00\times10^{-2}$$

$$v = \lambda f, \quad f = \frac{v}{\lambda}, \quad \omega = 2\pi f = \frac{2\pi v}{\lambda} = \frac{(2\pi)(30.0)}{0.5},$$

$$\omega = 377 \text{ rad/sec}$$

$$\therefore P = \frac{1}{2}(5.00\times10^{-2})(377)^2(0.1)^2(30.0)$$

$$= 1.07\times10^3 \text{ watts} = 1.07 \text{ Kwatt}$$

39 A two-dimensional water wave spreads in circular ripples. Show that the amplitude A at a distance r from the initial disturbance is proportional to $1/\sqrt{r}$. (*Suggestion*: Consider the energy carried by one outward-moving ripple.)

In the analysis of energy of a segment of a wave,

$$dU_x = \frac{1}{2}\mu\omega^2 A^2 \sin^2(kx-\omega t)\,dx$$

similarly,

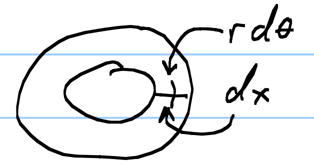
$$dK_x = \frac{1}{2}\mu\omega^2 A^2 \cos^2(kx-\omega t)\,dx$$

$$\therefore dE_x = dU_x + dK_x = \frac{1}{2}\mu\omega^2 A^2\,dx$$

Spread over a circular area, the total energy

$dE_T = dE_x \, r d\theta$, so over θ from $0 \rightarrow 2\pi$,

$dE_T = dE_x (2\pi r)$



$\therefore dE_T \propto A^2 (2\pi r)$, or $A \propto \sqrt{\frac{dE_T}{2\pi r}}$

Assuming energy of a circular wave stays the same, as r increases, A^2 will decrease, and as shown above, $A \propto \frac{1}{\sqrt{r}}$

40, Transverse waves are being generated on a rope under constant tension. By what factor is the required power increased or decreased if (a) the length of the rope is doubled and the angular frequency remains constant, (b) the amplitude is doubled and the angular frequency is halved, (c) both the wavelength and the amplitude are doubled, and (d) both the length of the rope and the wavelength are halved?

$v = \sqrt{\frac{T}{\mu}}$, $P = \frac{1}{2} \mu \omega^2 A^2 v$

(a) Assume mass also doubled $\Rightarrow \mu$ constant,
$\therefore v$ constant, P constant

(b) $\therefore \omega^2 A^2 = \frac{\omega^2}{4} (4A^2)$, so P constant

(c) No mention of frequency. Assume f constant, so $\omega = 2\pi f$ is unchanged. $\therefore v = \lambda f$, so v doubled. $\therefore P \propto A^2 v = (2A)^2 (2v) = 8A^2 v$, so P increased by 8.

If f halved, then v constant, so $\omega^2 A^2 \propto f^2 A^2$, so P unchanged.

(d) Assume μ unchanged.

Assume f unchanged.

$\therefore V$ is halved. $\therefore P$ is halved.

if f is doubled so v unchanged, then ω doubled, so P quadrupled.

42. It is found that a 6.00-m segment of a long string contains four complete waves and has a mass of 180 g. The string is vibrating sinusoidally with a frequency of 50.0 Hz and a peak-to-valley distance of 15.0 cm. (The "peak-to-valley" distance is the vertical distance from the farthest positive position to the farthest negative position.) (a) Write the function that describes this wave traveling in the positive x direction. (b) Determine the power being supplied to the string.

(a) $A = 7.5\text{ cm} = 0.075\text{ m}$

$K = \frac{2\pi}{\lambda}$, $\lambda = \frac{6.0}{4}$, so $K = \frac{8\pi}{6} = \frac{4}{3}\pi$

$\omega = 2\pi f = 100\pi$

$\therefore y(x,t) = 0.075 \sin\left(\frac{4}{3}\pi x - 100\pi t\right)$

$= 0.075 \sin(4.19x - 314t)$

(b) $P = \frac{1}{2}\mu\omega^2 A^2 v$, $\mu = \frac{0.180}{6.0} = 0.03$

$v = \lambda f = \frac{6.0}{4}(50.0) = 75.0$

$\therefore P = \frac{1}{2}(0.03)(100\pi)^2(0.075)^2(75.0)$

$= 625$ watts

43. A sinusoidal wave on a string is described by the equation

$$y = (0.15 \text{ m}) \sin(0.80x - 50t)$$

where x and y are in meters and t is in seconds. If the mass per unit length of this string is 12.0 g/m, determine (a) the speed of the wave, (b) the wavelength, (c) the frequency, and (d) the power transmitted to the wave.

$K = \frac{2\pi}{\lambda} = 0.80$, $\lambda = \frac{2\pi}{0.8} = 7.85$ m

$\omega = 50 = 2\pi f$, $\therefore f = 7.96$ Hz

$\therefore v = \lambda f = (7.96)(7.85) = 62.5$ m/sec.

(a) 62.5 m/sec

(b) 7.85 m

(c) 7.96 Hz

(d) $P = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}(0.012)(50)^2(0.15)^2(62.5)$

$= 21.1$ watts

44. A horizontal string can transmit a maximum power $\mathcal{P}_0$ (without breaking) if a wave with amplitude A and angular frequency ω is traveling along it. In order to increase this maximum power, a student folds the string and uses this "double string" as a medium. Determine the maximum power that can be transmitted along the "double string," assuming that the tension is constant.

Basically, μ is doubled.

$P = \frac{1}{2}\mu\omega^2A^2v$, $v = \sqrt{\frac{T}{\mu}} = \frac{\sqrt{T\mu}}{\mu}$

$\therefore P = \frac{1}{2}\omega^2A^2\sqrt{T\mu}$

$\therefore P$ increased by $\sqrt{2}$.

58. A wire of density ρ is tapered so that its cross-sectional area varies with x according to

$$A = (1.0 \times 10^{-3}\,x + 0.010)\ \text{cm}^2$$

(a) If the wire is subject to a tension T, derive a relationship for the speed of a wave as a function of position. (b) **What If?** If the wire is aluminum and is subject to a tension of 24.0 N, determine the speed at the origin and at $x = 10.0$ m.

(a) In the derivation of $v = \sqrt{\frac{T}{\mu}}$, $\Delta m = \mu\,\Delta s$,

where Δm is the mass of segment Δs. If Δs is small enough, the tapering of the wire is not important, but v refers to a point on the wire, so

$v = \sqrt{\frac{T}{\mu(x)}}$, where $\mu(x) = \mu$ at point x.

The tension along the wire will be the same.

$\therefore$ Need expression of $\mu(x)$

$$\mu(x) = \frac{\Delta m(x)}{\Delta x}, \quad \text{But } \Delta m(x) = \rho \cdot \Delta V(x)$$
$$\text{and } V(x) = A(x) \cdot \Delta x$$

$$\therefore \mu(x) = \frac{\rho \cdot A(x) \cdot \Delta x}{\Delta x} = \rho \cdot A(x)$$

$$\therefore \text{Velocity} = \sqrt{\frac{T}{\rho \cdot A(x)}}$$

(b) $\rho_{Al} = 2.7 \times 10^3 \text{ kg/m}^3$

$A(0) = 0.010 \text{ cm}^2 = 1.0 \times 10^{-2} \text{ cm}^2 = 1.0 \times 10^{-6} \text{ m}^2$

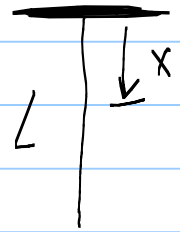
$A(10.0\text{ m}) = (1 + 0.010) \text{ cm}^2 = 1.01 \times 10^{-4} \text{ m}^2$

$$\therefore V_0 = \sqrt{\frac{24.0}{(2.7 \times 10^3)(1.0 \times 10^{-6})}} = 94.3 \text{ m/sec}$$

$$V_{10.0} = \sqrt{\frac{24.0}{(2.7 \times 10^3)(1.01 \times 10^{-4})}} = 9.38 \text{ m/sec}$$

59. A rope of total mass m and length L is suspended vertically. Show that a transverse pulse travels the length of the rope in a time interval $\Delta t = 2\sqrt{L/g}$. (*Suggestion:* First find an expression for the wave speed at any point a distance x from the lower end by considering the tension in the rope as resulting from the weight of the segment below that point.)

Unlike #58, here $v(x) = \sqrt{\dfrac{T(x)}{\mu}}$, $\mu = \dfrac{m}{L}$



$\therefore$ at point x, mass below x is

$m - m\left(\frac{x}{L}\right)$ = total mass − mass above x

$$T(x) = \text{weight of mass below } x$$

$$= mg\left(1 - \frac{x}{L}\right) = mg\left(\frac{L-x}{L}\right)$$

$$\therefore V(x) = \sqrt{\frac{mg\left(\frac{L-x}{L}\right)}{m/L}} = \sqrt{g(L-x)}$$

$$\therefore \Delta t = \frac{\Delta x}{v(x)} = \frac{\Delta x}{\sqrt{g(L-x)}},$$ x is within interval Δx

$$\therefore \text{Total time} = \int_0^L \frac{dx}{\sqrt{g(L-x)}}$$

Substitute $y = L - x$, $dy = -dx$,

$$\therefore -\int_L^0 \frac{dy}{\sqrt{gy}} = \int_0^L \frac{dy}{\sqrt{g}\sqrt{y}} = \frac{1}{\sqrt{g}}\int_0^L \frac{dy}{\sqrt{y}}$$

$$= \frac{1}{\sqrt{g}}\int_0^L y^{-\frac{1}{2}}\,dy = \frac{1}{\sqrt{g}}\cdot 2y^{\frac{1}{2}}\Big|_0^L$$

$$= \frac{1}{\sqrt{g}}\left(2\sqrt{L}\right) = 2\sqrt{\frac{L}{g}}$$

60. If an object of mass M is suspended from the bottom of the rope in Problem 59, (a) show that the time interval for a transverse pulse to travel the length of the rope is

$$\Delta t = 2\sqrt{\frac{L}{mg}}\left(\sqrt{M+m} - \sqrt{M}\right)$$

What If? (b) Show that this reduces to the result of Problem 59 when $M = 0$. (c) Show that for $m \ll M$, the expression in part (a) reduces to

$$\Delta t = \sqrt{\frac{mL}{Mg}}$$

(a) From #59, $T(x) = mg\left(\frac{L-x}{L}\right) + Mg$

$$\therefore V(x) = \sqrt{\frac{T(x)}{\mu}} = \sqrt{\frac{mg\left(\frac{L-x}{L}\right) + Mg}{m/L}}$$

$$= \sqrt{g(L-x) + \frac{M}{m}gL}$$

$$= \sqrt{gL\left(1+\frac{M}{m}\right) - gx}$$

$$\therefore \Delta t = \frac{\Delta x}{V(x)} = \frac{\Delta x}{\sqrt{gL\left(1+\frac{M}{m}\right)-gx}}$$

$$\therefore \text{Total time} = \frac{1}{\sqrt{g}} \int_0^L \frac{dx}{\sqrt{L\left(1+\frac{M}{m}\right)-x}}$$

Substitute $y = L\left(1+\frac{M}{m}\right) - x, \quad dy = -dx$

$x = 0 \Rightarrow y = L\left(1+\frac{M}{m}\right)$

$x = L \Rightarrow y = L\frac{M}{m}$

$$\therefore \frac{1}{\sqrt{g}} \int_{L\frac{M}{m}}^{L\left(1+\frac{M}{m}\right)} y^{-\frac{1}{2}}\, dy = \frac{1}{\sqrt{g}} \left[2y^{\frac{1}{2}}\right]_{L\frac{M}{m}}^{L\left(1+\frac{M}{m}\right)}$$

$$= \frac{2}{\sqrt{g}} \left[\sqrt{L\left(1+\frac{M}{m}\right)} - \sqrt{L\frac{M}{m}}\right]$$

$$= 2\sqrt{\frac{L}{g}} \left[\sqrt{\frac{m+M}{m}} - \sqrt{\frac{M}{m}}\right]$$

$$= 2\sqrt{\frac{L}{gm}} \left[\sqrt{M+m} - \sqrt{M}\right]$$

(b) Clearly, as $M \to 0$, the expression in

$$\text{(a)} \to 2\sqrt{\frac{L}{gm}}\left[\sqrt{m}\right] = 2\sqrt{\frac{L}{g}}$$

(c) If $m << M$, then $T(x) \approx Mg$

$$\therefore v = \sqrt{\frac{Mg}{m/L}} = \sqrt{\frac{MgL}{m}}$$

$$\therefore \Delta t = \frac{L}{v} = \frac{L}{\sqrt{\frac{MgL}{m}}} = L\sqrt{\frac{m}{MgL}}$$

$$= \underline{\underline{\sqrt{\frac{mL}{Mg}}}}$$

61. It is stated in Problem 59 that a pulse travels from the bottom to the top of a hanging rope of length L in a time interval $\Delta t = 2\sqrt{L/g}$. Use this result to answer the following questions. (It is not necessary to set up any new integrations.) (a) How long does it take for a pulse to travel halfway up the rope? Give your answer as a fraction of the quantity $2\sqrt{L/g}$. (b) A pulse starts traveling up the rope. How far has it traveled after a time interval $\sqrt{L/g}$?

Tension near bottom is small, as weight of rope segment at bottom is small. Tension high near ceiling as much rope is below.

$v = \sqrt{\frac{T}{\mu}}$, so, speed of wave is small at bottom,
speed is great at top.

From the derivation, the time for a wave to travel from bottom to a distance d above the bottom of the rope is $2\sqrt{d/g}$

(a) $\therefore$ To go from bottom to $L/2$ from bottom,

$$\Delta t = 2\sqrt{\frac{L/2}{g}} = \underline{\sqrt{\frac{2L}{g}}}$$

(b) $\sqrt{\frac{L}{g}} = 2\sqrt{\frac{d}{g}}$, $\frac{L}{g} = 4\frac{d}{g}$, $d = \frac{L}{4}$

$\therefore \frac{1}{4}$ way up the rope.