6.2 The Mobius Inversion Formula

Note Title 7/4/2005

1. (a). For each positive integer n, show

$$\mu(n)\mu(n+1)\mu(n+2)\mu(n+3)=0$$

Pf: From the Division Algorithm, let
$n=4a+b$, where $0 \le b < 4$
If $b=0$, Then $4|n \Rightarrow 2^2|n$ so $\mu(n)=0$
If $b=1$, Then $n+3=4a+4$, so $4|n+3$,
so $\mu(n+3)=0$
If $b=2$, $n+2=4a+4$, so $4|n+2 \Rightarrow \mu(n+2)=0$
If $b=3$, $n+1=4a+4$, so $4|n+1 \Rightarrow \mu(n+1)=0$

$\therefore$ for any n, at least one factor will yield $\mu=0$.

(b). For any integer $n \ge 3$, show $\sum_{k=1}^{n} \mu(k!)=1$

Pf: $\mu(4)=0$ since $4=2^2$.

If $n \ge 4$, Then $n!$ will contain 4 as a factor.
μ is multiplicative, so for $n \ge 4$,
$\mu(n!)=\mu(n)\cdots\mu(4)\mu(3)\mu(2)\mu(1)=0.$

$\therefore$ Only need to consider cases of $n=3$

$\mu(1)=1,\ \mu(2)=-1,\ \mu(3)=-1.$

$$\therefore \sum_{k=1}^{3}\mu(k!) = \mu(1!)+\mu(2!)+\mu(3!)$$
$$= \mu(1)+\mu(2)+\mu(6)$$
$$= 1+(-1)+1 = 1.$$

2. The Mangoldt function Λ is defined by

$$\Lambda(n)=\begin{cases}\log(p), & \text{if } n=p^k,\ p \text{ prime},\ k\geq 1\\ 0 & \text{otherwise}\end{cases}$$

Prove $\Lambda(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)\log(d) = -\sum_{d|n}\mu(d)\log(d)$

pf: Let $n=p^k$

(a) $\therefore \sum_{d|n}\mu\left(\frac{n}{d}\right)\log(d) = \mu(p^k)\log(1)$
$+\ \mu(p^{k-1})\log(p)$
$+\ \cdots$
$+\ \cdots\mu(p^{k-i})\log(p^i)$
$+\ \cdots$
$+\ \mu(p^0)\log(p^k)$

If $k=1$, The sum is $\mu(p^1)\log(1)+\mu(p^0)\log(p^1)$
$$= \mu(1)\log(p) = \log(p)$$

If $k>1$, $\mu(p^{k-i})=0$ except for $i=1,2$, and then the sum is The same as for $k=1$.

$$\therefore \sum_{d|n} \mu\left(\frac{n}{d}\right)\log(d) = \log(p) = \Lambda(n)$$

(b) $$\sum_{d|n} \mu(d)\log(d) = \mu(p^0)\log(1) + \mu(p^1)\log(p^1) + \cdots + \mu(p^i)\log(p^i) \vdots + \mu(p^k)\log(p^k)$$

Because $\mu(p^k)=0$ for $k>1$, The above sum reduces to, for all k,

$$\mu(p^0)\log(1) + \mu(p)\log(p) = -\log(p)$$

$$\therefore \sum_{d|n} \mu(d) \log(d) = -\Lambda(n)$$

3. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ for $n > 1$. If f is a multiplicative function not identically 0, prove that

$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_r))$$

Pf: Since μ and f are multiplicative, then μf is multiplicative (prob. #19, Sec. 6.1).

$\therefore$ By Th. 6.4, $F(n) = \sum_{d|n} \mu(d) f(d)$ is multiplicative. $\therefore$ If prove for $F(p^k)$ then, since $F(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = F(p_1^{k_1}) \cdots F(p_r^{k_r})$, will have proven for $F(n)$.

$\therefore$ Consider $F(p^k) = \sum_{d|p^k} \mu(d) f(d)$

$$= \mu(1) f(1) + \mu(p) f(p) + \cdots + \mu(p^k) f(p^k)$$

$= \mu(1) f(1) + \mu(p) f(p) \quad [\mu(p^i) = 0 \text{ for } i \geq 2]$

$= 1 \cdot f(1) + (-1) \cdot f(p) = 1 - f(p)$

since, for a multiplicative function not identically zero, $f(1) = 1$ (see Sec. 6.1).

$\therefore F(p^k) = 1 - f(p)$.

$\therefore \sum_{d|n} \mu(d) f(d) = (1 - f(p_1)) \cdots (1 - f(p_r))$

4. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. Establish the following:

(a) $\sum_{d|n} \mu(d) \tau(d) = (-1)^r$

Pf: By Prob. #3 above,

$\sum_{d|n} \mu(d) \tau(d) = [1 - \tau(p_1)] \cdot [1 - \tau(p_2)] \cdots [1 - \tau(p_r)]$

But $\tau(p) = 2$, for any prime p. $\therefore 1 - \tau(p) = -1$.

$\therefore \sum_{d|n} \mu(d) \tau(d) = (-1)^r$

(b) $\sum_{d|n} \mu(d)\sigma(d) = (-1)^r p_1 p_2 \cdots p_r$

Pf: By Prob. #3 above,

$$\sum_{d|n} \mu(d)\sigma(d) = [1-\sigma(p_1)][1-\sigma(p_2)]\cdots[1-\sigma(p_r)]$$

But $\sigma(p) = 1+p$ for any prime p.

$\therefore 1-\sigma(p) = -p$

$$\therefore \sum_{d|n} \mu(d)\sigma(d) = (-p_1)(-p_2)\cdots(-p_r)$$

$$= (-1)^r p_1 p_2 \cdots p_r$$

(c) $\sum_{d|n} \mu(d)/d = (1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_r})$

Pf: First, $f(n) = \frac{1}{n}$ is multiplicative since $f(mn) = \frac{1}{mn} = \frac{1}{m}\cdot\frac{1}{n} = f(m)f(n)$.

$\therefore$ By Prob. #3 above, where $f(n) = \frac{1}{n}$

$$\sum_{d|n} \mu(d)\frac{1}{d} = (1-f(p_1))\cdots(1-f(p_r)) = (1-\frac{1}{p_1})\cdots(1-\frac{1}{p_r})$$

(d) $\sum_{d|n} d\,\mu(d) = (1-p_1)(1-p_2)\cdots(1-p_r)$

Pf: Let $f(n) = n$. f is clearly multiplicative.
$\therefore$ By Prob. #3 above,

$$\sum_{d|n} d\,\mu(d) = \sum_{d|n} f(d)\mu(d) = (1-f(p_1))\cdots(1-f(p_r))$$
$$= (1-p_1)\cdots(1-p_r)$$

5. Let $S(n)$ denote the number of square-free divisors of n. Establish that

$$S(n) = \sum_{d|n} |\mu(d)| = 2^{\omega(n)}, \text{ where}$$

$\omega(n)$ = number of distinct prime divisors of n.

Pf: Note that
$$|\mu(n)| = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } p^2|n,\ p \text{ prime} \\ 1 & \text{if } n = p_1\cdots p_r,\ p_i \text{ distinct} \end{cases}$$

Let $f(n) = |\mu(n)|$. Let m, n be relatively prime.
Clearly, $f(1) = 1$
If $m=1$, $f(mn) = f(n) = f(m)f(n)$
Let m, n be relatively prime.

If $p^2 | m$, then $p^2 | mn$. $\therefore f(mn) = 0$ and $f(m) = 0$. $\therefore f(mn) = f(m) f(n)$

$\therefore$ Assume both m, n are square-free.

Let $m = p_1 \cdots p_r$, $n = q_1 \cdots q_s$. $p_i \neq q_i$ since $\gcd(m, n) = 1$. Clearly, $f(m) = 1$, $f(n) = 1$, and $f(mn) = 1$. $\therefore f(mn) = f(m) f(n)$.

$\therefore |\mu(n)|$ is multiplicative.

$\therefore$ By Th. 6.4, $\sum_{d|n} |\mu(n)|$ is multiplicative

$\therefore S(n)$ is multiplicative.

Consider $n = p^k$. The divisors of n are $1, p, p^2, \ldots p^k$.

$$\therefore \sum_{d|n} |\mu(n)| = |\mu(1)| + |\mu(p)| + |\mu(p^2)| + \cdots + |\mu(p^k)|$$

$$= 1 + 1 + 0 + \cdots + 0 = 2$$

The number of square-free diviors of p^k is 2 and is defined by $\sum_{d|n} |\mu(n)|$

Consider $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. From Th. 6.1,

all the square-free divisors of n are represented by $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, $0 \le a_i \le 1$

Since the number of square-free divisors from p_1 is 2 (1 and p_1); from p_2 is 2, ... from p_r is 2, the the total number

of square-free divisors is 2^r, or $2^{\omega(n)}$,

where $\omega(n) = r = \#$ of distinct prime divisors of n.

$$\therefore S(n) = S(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = S(p_1^{k_1}) S(p_2^{k_2}) \cdots S(p_r^{k_r})$$

$$= \left(\sum_{d \mid p_1^{k_1}} |\mu(p_1^{k_1})| \right) \cdots \left(\sum_{d \mid p_r^{k_r}} |\mu(p_r^{k_r})| \right)$$

$$= (2) \cdots (2) = 2^r = 2^{\omega(n)}$$

6. Find formulas for $\sum\limits_{d \mid n} \dfrac{\mu^2(n)}{\tau(n)}$ and $\sum\limits_{d \mid n} \dfrac{\mu^2(n)}{\sigma(n)}$ in terms of the prime factorization of n.

From Prob. # 19, Sec. 6.1, $\frac{\mu^2(n)}{\tau(n)}$ and $\frac{\mu^2(n)}{\sigma(n)}$ are both multiplicative.

$\therefore$ First consider case for $n = p^k$

$$\sum_{d|n} \frac{\mu^2(n)}{\tau(n)} = \frac{\mu^2(1)}{\tau(1)} + \frac{\mu^2(p)}{\tau(p)} + \frac{\mu^2(p^2)}{\tau(p^2)} + \cdots + \frac{\mu^2(p^k)}{\tau(p^k)}$$

$$= \frac{1}{1} + \frac{1}{2} + 0 + \cdots + 0$$

$$= \frac{3}{2}$$

$$\sum_{d|n} \frac{\mu^2(n)}{\sigma(n)} = \frac{\mu^2(1)}{\sigma(1)} + \frac{\mu^2(p)}{\sigma(p)} + \frac{\mu^2(p^2)}{\sigma(p^2)} + \cdots + \frac{\mu^2(p^k)}{\sigma(p^k)}$$

$$= \frac{1}{1} + \frac{1}{p+1} + 0 + \cdots 0$$

$$= \frac{p+2}{p+1}$$

$\therefore$ Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$,

$$F(n) = \sum_{d|n} \frac{\mu^2(n)}{\tau(n)}, \quad G(n) = \sum_{d|n} \frac{\mu^2(n)}{\sigma(n)}$$

Both F and G are multiplicative,

$$\therefore F(n) = F(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = F(p_1^{k_1}) F(p_2^{k_2}) \cdots F(p_r^{k_r})$$

and $$G(n) = G(p_1^{k_1}) G(p_2^{k_2}) \cdots G(p_r^{k_r})$$

$$\therefore \sum_{d|n} \frac{\mu^2(n)}{\tau(n)} = F(p_1^{k_1}) \cdots F(p_r^{k_r}) = \left(\frac{3}{2}\right) \cdots \left(\frac{3}{2}\right)$$

$$= \left(\frac{3}{2}\right)^r, \quad r = \#\text{ distinct primes.}$$

$$\sum_{d|n} \frac{\mu^2(n)}{\sigma(n)} = G(p_1^{k_1}) \cdots G(p_r^{k_r})$$

$$= \left(\frac{p_1+2}{p_1+1}\right)\left(\frac{p_2+2}{p_2+1}\right) \cdots \left(\frac{p_r+2}{p_r+1}\right)$$

7. The Liouville λ-function is defined by:

$$\lambda(1) = 1$$
$$\lambda(n) = (-1)^{k_1+k_2+\cdots+k_r}, \quad n > 1,\ n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

(a) Prove λ is a multiplicative function

Pf: Let m, n be relatively prime,

$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, $n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$

$mn = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}$, where $p_i \neq q_j$

since $\gcd(m,n) = 1$.

$$\therefore \lambda(mn) = (-1)^{k_1 + \cdots + k_r + j_1 + \cdots j_s}$$
$$= (-1)^{k_1 + \cdots + k_r} \cdot (-1)^{j_1 + \cdots + j_s}$$
$$= \lambda(m) \cdot \lambda(n)$$

(b). Given $n > 0$, verify

$$\sum_{d \mid n} \lambda(d) = \begin{cases} 1 & \text{if } n = m^2, \text{ some } m \\ 0 & \text{otherwise} \end{cases}$$

Pf: Let $F(n) = \sum_{d \mid n} \lambda(d)$. F is multiplicative by Th. 6.4. Plan is to prove for $n = p^k$, then extrapolate.

Let $n = p^k$.

$$\therefore F(n) = \lambda(1) + \lambda(p) + \cdots + \lambda(p^k)$$
$$= 1 + (-1) + (-1)^2 + (-1)^3 + \cdots + (-1)^{k-1} + (-1)^k$$

If k is even, $n = p^{2w}$, where $k = 2w$.
$\therefore$ let $m = p^w$, $\therefore$ $n = m^2$
Also, $F(n) = 1$

If k is odd, $F(p^k) = 0$

$\therefore$ Now let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$\therefore$ $F(n) = F(p_1^{k_1}) \cdots F(p_r^{k_r})$

If $n = m^2$ for some m, then all the k_i are even, so $F(p_i^{k_i}) = 1$ from above.
$\therefore$ $F(n) = 1$

If any one of the k_i is odd, then $F(p_i^{k_i}) = 0$, so $F(n) = 0$.

8. For any integer $n \geq 1$, verify formulas below:

(a) $\sum_{d|n} \mu(d)\lambda(d) = 2^{\omega(n)}$, $\omega(n) = \#$ distinct prime divisors of n

Pf: $\mu \cdot \lambda$ is multiplicative (Prob. 19, Sec. 6.1)

$\therefore$ Consider $n = p^k$

$$\therefore \sum_{d|n} \mu(d)\lambda(d) = \mu(1)\lambda(1)$$

$$+ \mu(p)\lambda(p) + \cdots + \mu(p^k)\lambda(p^k)$$

$$= 1\cdot 1 + (-1)(-1) + \cdots + 0\cdot(-1)^k$$

$$= 2$$

That is, for $n=p^k$, $F(n) = \sum_{d|n} \mu(d)\lambda(d) = 2$

$\therefore$ for $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$,

$$F(n) = F(p_1^{k_1}) F(p_2^{k_2}) \cdots F(p_r^{k_r})$$

$$= 2\cdot 2 \cdots 2 = 2^r = 2^{\omega(n)}$$

since $\omega(n) = r$.

(b) $\sum_{d|n} \lambda\left(\frac{n}{d}\right) 2^{\omega(d)} = 1$

Pf: Lemma: If $f(n)$, $g(n)$ are multiplicative, Then so is $F(n) = \sum_{d|n} f\left(\frac{n}{d}\right)\cdot g(d)$

Pf: Let m, n be relatively prime

positive integers.

$$\therefore F(mn) = \sum_{d \mid mn} f\left(\frac{mn}{d}\right) \cdot g(d) =$$

$$\sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f\left(\frac{mn}{d_1 d_2}\right) \cdot g(d_1 d_2) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f\left(\frac{m}{d_1}\right) f\left(\frac{n}{d_2}\right) g(d_1) g(d_2)$$

$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f\left(\frac{m}{d_1}\right) g(d_1) f\left(\frac{n}{d_2}\right) g(d_2)$$

$$= \left(\sum_{d_1 \mid m} f\left(\frac{m}{d_1}\right) g(d_1)\right)\left(\sum_{d_2 \mid n} f\left(\frac{n}{d_2}\right) g(d_2)\right) = F(m) F(n)$$

$\therefore F(n) = \sum_{d \mid n} \lambda\left(\frac{n}{d}\right) 2^{\omega(d)}$ is multiplicative. by above Lemma and problems 19, 20(a), Sec. 6.1

$\therefore$ Consider $n = p^k$

$$F(p^k) = \sum_{d \mid p^k} \lambda\left(\frac{p^k}{d}\right) 2^{\omega(d)}$$

$$= \lambda\left(\frac{p^k}{1}\right) 2^{\omega(1)} + \lambda\left(\frac{p^k}{p}\right) 2^{\omega(p)} + \ldots + \lambda\left(\frac{p^k}{p^{k-1}}\right) 2^{\omega(p^{k-1})} + \lambda\left(\frac{p^k}{p^k}\right) 2^{\omega(p^k)}$$

$$= (-1)^k \cdot 1 + (-1)^{k-1} \cdot 2 + \dots + (-1)^1 \cdot 2 + 1 \cdot 2$$

There are k terms of $(-1)^{k-1} \cdot 2 + \dots + (-1)^1 \cdot 2 + 1 \cdot 2$

$\therefore$ If k is even, $(-1)^k \cdot 1 = 1$, and

$$\underbrace{\left[(-1)^{k-1} + (-1)^{k-2} + \dots + (-1)^1 + 1\right]}_{k = \text{even \# terms, alternating signs}} \cdot 2 = 0 \cdot 2 = 0$$

$$\therefore F(p^k) = 1 + 0 = 1$$

If k is odd, $(-1)^k \cdot 1 = -1$, and

$$\left[\underbrace{(-1)^{k-1} + (-1)^{k-2} + \dots + (-1)^1}_{k-1 = \text{even \# terms, alternating signs}} + 1\right] \cdot 2 = [0 + 1] \cdot 2 = 2$$

$$\therefore F(p^k) = (-1) + 2 = 1$$

$\therefore$ If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$,

$$F(n) = F(p_1^{k_1}) F(p_2^{k_2}) \cdots F(p_r^{k_r}) = 1 \cdot 1 \cdots 1 = 1$$